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ON THE SECOND HOMOLOGY GROUP OF THE DISCRETE GROUP OF DIFFEOMORPHISMS OF THE CIRCLE

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Let G be the discrete group of orientation preserving diffeomorphisms of the circle, and let H be the subgroup of diffeomorphisms which are the identity in a neighborhood of a given point. It is proved that there is a short exact sequence in the integral homology of groups $0 \longrightarrow H_2(H) \longrightarrow H_2(G) \xrightarrow{E} Z \longrightarrow 0$. The epimorphism E can be identified with the Euler class for transversally foliated circle bundles. The sequence has a natural interpretation in terms of cobordism classes of foliated bundles with Euler class equal to 0.

Introduction

Let G be the discrete group of orientation preserving diffeomorphisms of the circle C and let H be the subgroup of diffeomorphisms which are the identity in a neighborhood of a given point. The object of this paper is to prove that there is a short exact sequence in the integral homology of groups (Theorem 3)

$$0 \longrightarrow H_2(H) \longrightarrow H_2(G) \xrightarrow{E} Z \longrightarrow 0.$$

This short exact sequence has a direct geometrical application to foliated bundles which will be explained in Section 4. In this setting the epimorphism E of the sequence can be identified with the Euler class for flat circle bundles. This is done explicitly in [7] where the constructions of this paper are also applied to obtain a formula for E intrinsic to the bar construction of G, and an upper bound for E on any CW complex in terms of its 2-dimensional simplicial structure.

To derive the short exact sequence I will apply the classical construction of a group acting on a set which results in a spectral sequence converging to the homology of the group. I will specifically consider the case when a subgroup S of the group G is given and G acts on the cosets G/S. The associated spectral sequence is well known and yields a generalization of the Hochschild–Serre spectral sequence.

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In Section 1, I describe the spectral sequence. A group operating on a set gives rise to a simplicial groupoid, extending by nerves gives a bisimplicial set, and the homology of the bisimplicial set is the homology of the group. Making a group action into a groupoid is an old construction which dates back to work of Ehresmann and Reidemeister. Simplicial groupoids are also familiar objects; see [4] and [5] for expositions. The interested reader should also refer to [1] and [2] for more on such constructions and an extensive bibliography.

Applications of this approach to foliations arise as follows. The classifying spaces for foliations, as is well known, can be considered to be the classifying spaces of certain topological groupoids. Simplicial groupoids arise naturally in this context, and their relationship to the simplicial groupoids arising from group actions can be exploited to obtain results comparing discrete groups of diffeomorphisms to foliations [6].

The difficulty in comparing G and S in many interesting cases lies in the 'component complex' π_* or 'orbit space' which is at the base of the E^1 term of the spectral sequence. The homology of the group $\mathrm{SL}(2,\mathbb{R})$ made discrete provides an analagous but more complicated example than the one considered here. It has been extensively studied. Sah and Parry [11] use the complex $\mathrm{SL}(2,\mathbb{R})/W$ which is basically $E^1_{p,0}=\pi_*$ for the subgroup S = matrices of the form $\left(\begin{smallmatrix} a & b \\ 0 & 1/a \end{smallmatrix}\right)$. The invariant in $H_3(\mathrm{SL}(2,\mathbb{R}))$ obtained using the dilogarithm arises from the nontransitivity of the action of $\mathrm{SL}(2,\mathbb{R})$ on S^1. In the case I will discuss, the Euler class E appears in the homology of π_*, (and it is the form of its appearance which gives rise to the formula for E which is the main result of [7]). The action is transitive, and the situation is easier. One of my objectives in proving Theorem 3 in the manner of this paper is to do an explicit calculation on the level of π_* in the hope of shedding some light on the more difficult examples.

1. A spectral sequence for a subgroup S of a discrete group G

Consider $S \subset G$ a subgroup S of a group G. Let G/S be the set of left cosets of S in G. For each $p \geq 0$ define a category $\{G/S\}_p$ as follows:

$$\text{Objects} \quad \{G/S\}_p = G/S \times \cdots \times G/S = (G/S)^{p+1},$$
$$\text{Morphisms} \quad \{G/S\}_p = G \times (G/S)^{p+1}.$$

Let $s(\alpha)$ and $t(\alpha)$ denote the source and target objects of a given morphism α. Define

$$s(g, x_0 S, \ldots, x_p S) = (x_0 S, \ldots, x_p S),$$
$$t(g, x_0 S, \ldots, x_p S) = (g x_0 S, \ldots, g x_p S).$$

Define the composition $\beta \circ \alpha$ of morphisms β and α as follows: if

$$\alpha = (f, x_0 S, \ldots, x_p S), \qquad \beta = (g, x_0 S, \ldots, x_p S), \qquad t(\alpha) = s(\beta),$$

then

$$\beta \circ \alpha = (gf, x_0 S, \ldots, x_p S).$$

Note that $\{G/S\}_p$ is a *groupoid*; that is, each morphism is invertible.

The groupoids $\{G/S\}_p$ fit together to form a *simplicial groupoid* $\{G/S\}_* = \bigcup_{p \geq 0} \{G/S\}_p$. Define

$$\partial_i (g, x_0 S, \ldots, x_p S) = (g, x_0 S, \ldots, \widehat{x_i S}, \ldots, x_p S),$$
$$s_i (g, x_0 S, \ldots, x_p S) = (g, x_0 S, \ldots, x_i S, x_i S, \ldots, x_p S).$$

These faces and degeneracies are defined on objects and morphisms for each $\{G/S\}_p$. For convenience, the objects of $\{G/S\}_p$ may be identified with the subset of identity morphisms.

Consider the simplicial set of objects of $\{G/S\}_*$: $\{G/S\}^* = \bigcup_{p \geq 0} (G/S)^{p+1}$. This complex has a contractible realization, for it is just an infinite simplex with vertices G/S.

Now form a bisimplicial set $\{G/S\}_{**}$ by extending by nerves in the vertical direction.

$$\{G/S\}_{**} : (p, q) \to N_q(\{G/S\}_p).$$

Realizing $\{G/S\}_{**}$ in the p-direction first yields a simplicial *space* whose space of q-simplices is $N_q G \times |\{G/S\}^*|$. Since $|\{G/S\}^*|$ is contractible, the following holds:

Theorem 1. $|\{G/S\}_{**}| \to |NG| = BG$ *is a homotopy equivalence.* □

Computing homology vertically gives a double abelian group and a spectral sequence converging to BG.

Theorem 2. $E^1_{p,q} = H_q(B\{G/S\}_p) \Rightarrow H_{p+q}(BG)$. □

In this theorem $B\{G/S\}_p$ is the classifying space of the discrete groupoid $\{G/S\}_p$.

To interpret and apply this result, recall that $B\{G/S\}_p$ (in fact, as is well known, the classifying space of any discrete groupoid) is a disjoint union of $K(\pi, 1)$'s. To describe the fundamental group π of a given component choose a base point, equivalently an object, in each component. Then π is the group of morphisms of $\{G/S\}_p$ where source and target is the given object. In other words π is the isotropy group S_x of the object $x \in (G/S)^{p+1}$. To compute S_x, suppose $x = (x_0 S, \ldots, x_p S)$ is a chosen base point. Then $S_x = S \cap (x_0 S x_0^{-1}) \cap \cdots \cap (x_p S x_p^{-1})$. From this point on I will write $H_*(G)$ for the homology of BG when G is discrete.

The above bisimplicial set and its associated spectral sequence is a slight reformulation of a classical constuction on group actions.

Suppose G acts on a set X. Form the groupoid $G \times X$ as above. Let EX be the simpicial set $n \to EX_n = X^{n+1}$ which has a contractible realization ([12], or [6]). G acts on EX so the simplicial groupoid $G \times EX$ may be formed. So one obtains a bicomplex and a spectral sequence

$$E^1_{p,q} = H_q(G\times X^p) \Rightarrow H_{p+q}(G).$$

The components of $G\times X$ are the orbits of the action and the 'vertex' groups are the isotropy groups of the action.

Each formulation has its advantages: the classical version allows G to act on any set; the above version allows one to distinguish a subgroup which is to be the 'fiber'.

2. A short exact sequence for diffeomorphisms of the circle

Let G be the group of orientation preserving C^r diffeomorphisms of the circle $S^1=\mathbb{R}/Z$, $0\le r\le\infty$, $r\ne 2$. The reason for the restriction $r\ne 2$ will be discussed below. Let H be the subgroup of G consisting of those diffeomorphisms which are the identity in a neighborhood of $0\in\mathbb{R}/Z$. The main theorem of this section is

Theorem 3. *The inclusion $i: H\subset G$ gives rise to a short exact sequence $0\to H_2(H)\to H_2(G)\to Z\to 0$.* □

I will discuss the geometric significance of this sequence in Section 4.

The group $H_1(H)=H_{\mathrm{ab}}=H$ – made abelian, has been computed for $r\ne 2$. For finite r the result $H_1(H)=0$ is due to Mather [8]. For $r=\infty$, $H_1(H)=0$ is due to Moser [10] and Epstein [3]. The case $r=2$ is not solved; see [9] for a discussion of what happens in this case. Theorem 3 depends on that fact that $H_1(H)=0$ so I will restrict to the cases when this is known to be true.

3. Proof of Theorem 3

I will apply the spectral sequence of Theorem 2 to the group G of orientation preserving diffeomorphisms of the circle with *subgroup S of those diffeomorphisms which fix* $0\in\mathbb{R}/Z=C$. This seems to be easier than applying it directly to $H\subset G$.

I will compute the relevant terms of the spectral sequence, namely

(i) $E^2_{p,0}$, $0\le p\le 2$;

(ii) $E^2_{p,1}$, $0\le p\le 1$;

(iii) $E^2_{p,2}$, $p=1$,

and obtain $E^2_{2,0}\cong Z$, $E^2_{0,2}\cong H_2(S)$ and all other terms 0.

(i) $E^2_{p,0}$, $0\le p\le 2$. Consider again $\pi_*=\pi_0(B\{G/S\}_*)$. A component of $B\{G/S\}_p$ is an equivalence class $G/(x_0X,\dots,x_pS)$; the ith face map deletes x_iS and the ith degeneracy map doubles x_iS. In every equivalence class there is a representative of the form $(S,x_1S,\dots,x_pS)$. Using these representatives and deleting the first entry gives a 1-1 correspondence between components of $B\{G/S\}_p$ and $S/(x_1S,\dots,x_pS)$. Now

$$\partial_0(x_1S,\dots,x_pS) = (x_1^{-1}x_2S,\dots,x_1^{-1}x_pS),$$
$$\partial_i(x_1S,\dots,x_pS) = (x_1S,\dots,\widehat{x_iS},\dots,x_pS)$$

and

$$\sigma_i(x_1S, \dots, x_pS) = (x_1S, \dots, x_iS, x_iS, \dots, x_pS).$$

(Note that the formula for the 0th face map follows because $(S, x_1S, \dots, x_pS)$ must be rewritten with first entry S.)

Lemma 4.

$$E^2_{p,0} = H_p(|\pi_*|) = \begin{cases} 0, & p=1,3, \\ Z, & p=0,2. \end{cases}$$

Proof. If $x, y \in G$ and $x(0)=y(0)$, then $xS=yS$. So there is a 1-1 correspondence between cosets xS and points of the circle C: $x \to x(0)$. By the discussion preceding the statement of Lemma 4 there is a 1-1 correspondence of components of $B\{G/S\}_p$ and p-tuples of points of C, $(\alpha_1, \dots, \alpha_p)$, modulo the equivalence $(\alpha_1, \dots, \alpha_p) \sim (\beta_1, \dots, \beta_p)$ if there is a diffeomorphism $s \in S$ so that $s(\alpha_i)=\beta_i$ for all $1 \le i \le p$. The face maps on respresentatives are given by

$$\partial_0(\alpha_1, \dots, \alpha_p) = (\alpha_2 - \alpha_1, \dots, \alpha_p - \alpha_1),$$
$$\partial_i(\alpha_1, \dots, \alpha_p) = (\alpha_1, \dots, \hat{\alpha}_i, \dots, \alpha_p),$$
$$\sigma_i(\alpha_1, \dots, \alpha_p) = (\alpha_1, \dots, \alpha_i, \alpha_i, \dots, \alpha_p).$$

The calculations can be further simplified by observing that G acts simplicially on $(p+1)$-tuples of *distinct* cosets $(x_0S, \dots, x_pS)$, for the face maps of the simplicial groupoid $\{G/S\}_*$ of Section 1 preserve such elements. So one can form a subcomplex of $\{G/S\}_*$ by using just distinct cosets and adding on degeneracies to obtain a simplicial set. In computing homology the degenerate simplices can be set equal to 0.

Then in π_* there is one 0-simplex corresponding to the coset S; one non-degenerate 1-simplex corresponding to (S, xS), $x \neq 0$; in general the non-degenerate n-simplices are in 1-1 correspondence with the set of permutations of $(1, 2, \dots, n)$.

If C is viewed as $\mathbb{R}/(n+1)Z$, then the set of k-simplices of π_*, $k \le n$ is *actually* the set consisting of $(1, 2, \dots, k)$ together with all permutations of the entries. This notation makes it easier to compute the boundary on the chains of π_*,

$$\partial : C_k(\pi_*) \to C_{k-1}(\pi_*).$$

If $\alpha = (\alpha_1, \dots, \alpha_k)$ is a permutation of $(1, \dots, k)$, then

$$\partial_0 \alpha = (\alpha_2 - \alpha_1, \dots, \alpha_k - \alpha_1),$$
$$\partial_i \alpha = (\alpha_1, \dots, \hat{\alpha}_i, \dots, \alpha_k), \quad 1 < i < k_5$$

where *the entries of the right-hand side are taken* mod $(n+1)$.

Now for convenience in computing the 4-skeleton view C as $\mathbb{R}/5\mathbf{Z}$.

From $\partial(1,2)=(1)$ it follows that $H_1(|\pi_*|)=0$. Also, $\partial(2,1)=(1)$.

I claim that the cycle $z=(2,1)-(1,2)$ and all multiples of it do not bound and z generates $H_2(|\pi_*|) \cong Z$.

To see this, compute $\partial : C_3(\pi_*) \to C_2(\pi_*)$.

$$\partial(1,2,3) = (1,2)-(1,2)+(1,2)-(1,2) = 0,$$
$$\partial(2,1,3) = (2,1)-(1,2)+(1,2)-(2,1) = 0$$

and similar calculations mod 5 show that ∂ on all other generators of $C_3(\pi_*)$ is zero, $\partial : C_2(\pi_*) \to C_1(\pi_*)$ is an epimorphism $Z \oplus Z \to Z$, so the kernel, $H_2(|\pi_*|)$, is isomorphic to Z. Clearly, z is a generator.

The following calculations show that each generator of $C_3(\pi_*)$ bounds. Each such generator is already a cycle. Hence $H_3(|\pi_*|)=0$

$$\partial(1,2,3,4) = (1,2,3), \qquad \partial(4,3,2,1) = (3,2,1),$$
$$\partial(2,1,3,4) = (3,1,2), \qquad \partial(3,4,2,1) = (1,3,2),$$
$$\partial(2,3,4,1) = 2(1,2,3)-(2,3,1),$$
$$\partial(3,2,4,1) = (3,1,2)-(3,2,1)+(2,1,3).$$

This completes the proof of Lemma 4. $\square$

(ii) $E^2_{p,1}$, $0 \le p \le 1$. We first give a convenient description of the complex $E^1_{*,1}$.

Let H be the subgroup of S consisting of diffeomorphisms which are the identity in a neighborhood of $0 \in \mathbb{R}/Z$. H is a normal subgroup and its quotient is G_0, the germs of elements of S at 0.

$$1 \to H \to S \to G_0 \to 1.$$

Since $H_{ab}=0$, there is an isomorphism

$$H_1(S) \to H_1(G_0).$$

View C as $\mathbb{R}/(n+1)Z$. Consider again $\alpha = (\alpha_1, \dots, \alpha_p)$, a permutation of $(1,\dots,p)$ considered as an object of $\{G/S\}_p$. (Recall that 0 is implicitly the 1st entry of $(\alpha_1,\dots,\alpha_p)$.) Let S_α be the object

$$\alpha : S_\alpha = S \cap S_{\alpha_1} \cap \cdots \cap S_{\alpha_p}$$

where S_{α_i} is the subgroup of G keeping $\alpha_i \in \mathbb{R}/(n+1)Z$ fixed. Let H_α be those diffeomorphisms which are the identity in a neighborhood of 0 and each i. Clearly, $H_\alpha \cong H \times \cdots \times H$ $p+1$ times. The quotient G/H_α is $G_0 \times G_{\alpha_1} \times \cdots \times G_{\alpha_p}$ where G_{α_i} is the group of germs of elements of G at $\alpha_i \in \mathbb{R}/(n+1)Z$. Each G_{α_i} is isomorphic to G_0, so there is a short exact sequence

$$1 \to H_\alpha \to S_\alpha \to G_0 \times G_{\alpha_1} \times \cdots \times G_{\alpha_p} \to 1$$

or

$$1 \to H^{p+1} \to S_\alpha \to G_0^{p+1} \to 1.$$

Since $H_1(H^{p+1})=0$, there is an isomorphism

$$H_1(S_\alpha) \to H_1(G_0) \oplus \cdots \oplus H_1(G_{\alpha_p}) \cong [H_1(G_0)]^{p+1}.$$

This leads to the following description of $E^1_{p,1}$, $1 \le p \le n$. $E^1_{p,1}$ is the direct sum of $H_1(S_\alpha)$ taken over all permutations $\alpha = (\alpha_1, \dots, \alpha_p)$. An element of $E^1_{p,1}$ is a sum of terms of the form

$$[a_0, \dots, a_p] \cdot (\alpha_1, \dots, \alpha_p)$$

where $a_0 \in H_1(G_0)$, $a_i \in H_1(G_{\alpha_i})$ and α is a permutation of $(1, \dots, p)$.

Note that $[a_0, \dots, a_p] \cdot (\alpha_1, \dots, \alpha_p) + [b_0, \dots, b_p] \cdot (\alpha_1, \dots, \alpha_p) = [a_0 + b_0, \dots, a_p + b_p] \cdot (\alpha_1, \dots, \alpha_p)$.

The boundaries are as follows:

$$\partial [a_0, \dots, a_p] \cdot (\alpha_1, \dots, \alpha_p) = [a_0, \dots, \hat{a}_i, \dots, a_p] \cdot \partial_\alpha(\alpha_1, \dots, \alpha_p)$$

where ∂_α is the corresponding boundary on $C(\pi_*)$.

Lemma 5. $E^2_{p,1} = 0$, $p = 0, 1, 2$.

Proof. $p = 0$. $\partial : E^1_{1,1} \to E^1_{0,1}$ is given by

$$\partial [a, b] \cdot (1) = [b - a] \cdot (\text{point})$$

which is onto and the kernel consists of elements of the form $[a, a] \cdot (1)$. So $E^2_{0,1} = 0$.

$p = 1$. $\partial : E^1_{2,1} \to E^1_{1,1}$ is given by

$$\partial [a, b, c] \cdot (1, 2) = [b, c] \cdot (1) - [a, c] \cdot (1) + [a, b] \cdot (1) = [b, b] \cdot (1)$$

and

$$\partial [a, b, c] \cdot (2, 1) = [b, b] \cdot (1).$$

So $E^2_{1,1} = 0$ and a 2-cycle in the complex $E^1_{1,1}$ is a sum of cycles of the form $z = [a, b, c] \cdot (2, 1) - [a', b', c'] \cdot (1, 2)$ with $b = b'$.

$p = 2$. Let $z = [a, b, c] \cdot (2, 1) - [a', b, c'] \cdot (1, 2)$ as above. Consider $w = [a, b, c, d] \cdot (2, 3, 1) \in E^2_{3,1}$.

$$\partial w = [b - a, c - b, d - c] \cdot (1, 2) - [0, c - b, 0] \cdot (2, 1).$$

So z is homologous to

$$z' = [a, b, c] \cdot (2, 1) - [0, b, 0] \cdot (2, 1) = [a, 0, c] \cdot (2, 1).$$

Now consider $w' = [a, b, c, d] \cdot (3, 2, 1)$.

$$\partial w' = [b - a, 0, d - c] \cdot (2, 1)$$

so z' bounds and $E^2_{2,1} = 0$. This completes the proof of Lemma 5. $\square$

(iii) $E^2_{0,2}$. View C as $\mathbb{R}/Z$. Let $S_{1/2}$ be the subgroup of S keeping 0 and 1/2 fixed. Then $S_{1/2} = S \cap rSr^{-1}$ where r is rotation by 180° and this is the isotropy group of the object $(0, 1/2)$. Therefore,

$$E^2_{0,2} = H_2(S)/\partial_* H_2(S_{1/2})$$

and

$$\partial_* = (\partial_0)_* - (\partial_1)_*$$

where $(\partial_0)_* : H_2(S_{1/2}) \to H_2(S)$ is induced by the inclusion ∂_0 of $S_{1/2}$ in S and $(\partial_1)_* : H_2(S_{1/2}) \to H_2(S)$ is induced by conjugation, ∂_1, by a rotation of 180°. I will prove

Proposition 6. $H_2(S)/\partial_* H_2(S_{1/2}) \cong H_2(H)$.

Proof. Consider again $H \subset S$ and let $H_{1/2} \subset S_{1/2}$ be those diffeomorphisms which are the identity in a neighborhood of 0 and 1/2. Let $\bar{\partial}_0$ be the inclusion of $H_{1/2}$ in H and let $\bar{\partial}_1$ be the homomorphism $H_{1/2} \to H$ which first conjugates an element by a rotation of 180°, then includes it in H. There is a commutative diagram

$$\begin{tikzcd}
H_{1/2} \arrow[r, shift left, "\bar{\partial}_0"] \arrow[r, shift right, "\bar{\partial}_1"'] \arrow[d] & G_0 \arrow[d] \\
S_{1/2} \arrow[r, shift left, "\partial_0"] \arrow[r, shift right, "\partial_1"'] & S
\end{tikzcd}$$

where the vertical arrows are induced by inclusions.

Lemma 7. $(\bar{\partial}_0)_* = (\bar{\partial}_1)_* : H_2(H_{1/2}) \to H_2(H)$.

Proof. $H_{1/2} = H^+ \times H^-$ where H^+ is the subgroup of diffeomorphisms which are the identity in a neighborhood of $[0, 1/2]$ and H^- is the subgroup of diffeomorphisms which are the identity in a neighborhood of $[1/2, 1]$. So $(\bar{\partial}_0)_* - (\bar{\partial}_1)_*$ identifies with a homomorphism $H_2(H^+) \oplus H_2(H^-) \to H_2(H)$. Now any finite number of elements of H^+ are conjugate *via an element of* H to elements of H^- and any finite number of elements of $\bar{H}$ are conjugate via an element of H to elements of H^+; as a conjugating element one can choose a diffeomorphism which is rotation by 180° on a large enough interval in $C - \{0\}$. This implies that

$$(\bar{\partial}_0)_*(\alpha, 0) = (\bar{\partial}_1)_*(\alpha, 0)$$

and

$$(\bar{\partial}_0)_*(0, \beta) = (\bar{\partial}_1)_*(0, \beta),$$

for conjugation in H induces the identity on the homology of H. This completes the proof of the lemma. □

Lemma 8. $0 \to H_2(H) \to H_2(S) \to H_2(G_0) \to 0$ *is exact.*

Proof. Recall that G_0 is the group of germs of diffeomorphisms at $0 \in \mathbb{R}/Z$. There is a short exact sequence

$$1 \to H \to S \to G_0 \to 1$$

as well as a short exact sequence

$$1 \to H_{1/2} \to S_{1/2} \to G_0 \times G_0 \to 1$$

and $H_{1/2}$ is isomorphic to $H \times H$.

The spectral sequences obtained from these particular short exact sequences, together with the fact that $H_{ab} = 0$ lead to the following commutative diagram of exact sequences:

$$\begin{tikzcd}
H_3(G_0) \arrow[r] & H_2(H) \arrow[r] & H_2(S) \arrow[r] & H_2(G_0) \arrow[r] & 0 \\
H_3(G_0 \times G_0) \arrow[r] \arrow[u] & H_2(H \times H) \arrow[r] \arrow[u, "(\bar{\partial}_0)_* - (\bar{\partial}_1)_*"'] & H_2(S_{1/2}) \arrow[r] \arrow[u] & H_2(G_0 \times G_0) \arrow[r] \arrow[u, "(\partial_0)_* - (\partial_1)_*"'] & 0
\end{tikzcd}$$

Now the homomorphism $\partial_* = (\bar{\partial}_0)_* - (\bar{\partial}_1)_*$ is zero by the above lemma. On the other hand, it is easy to see that the vertical homomorphisms on the far ends of the sequence are onto: The homomorphism induced by ∂_0 on $H_*((G_0 \times G_0)) \to H_*(BG_0)$ is the homomorphism induced by projection on the second factor, and ∂_1 induces projection on the first factor. This implies that $H_3(G_0) \to H_2(H)$ is the 0-homomorphism, and this proves Lemma 8. $\square$

In fact a completely analogous proof shows that

$$0 \to H_2(H \times H) \to H_2(S_{1/2}) \to H_2(G_0 \times G_0) \to 0.$$

is exact.

Now $\partial_* : H_2(S_{1/2}) \to H_2(S)$ is easy to compute. On the right end of the above short exact sequence it is onto $H_2(G_0)$; on the left end it is the zero homomorphism. This completes the proof of Proposition 6 and hence Theorem 3. $\square$

4. Geometric remarks

Recall, BG classifies circle bundles with foliations transverse to the fibers and BS classifies line bundles with transverse foliations 'trivial' outside of a compact set in each fiber, see e.g. [14]. Thurston [13], has proved that $H_2(G)$ has a homomorphism onto the real numbers by explicitly constructing uncountably many distinct cobordism classes of foliated circle bundles. No analagous construction is known for $H_2(S)$. In [7] I show that the homomorphism $E : H_2(G) \to Z$ of this paper is the Euler class. So any element of $H_2(G)$ with Euler class zero can be uniquely pulled back to $H_2(S)$. Perhaps it is possible to give an explicit construction of this pullback using the spectral sequence of this paper, and to apply it to the Thurston examples.

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